

Avoiding Similarity Transformations in the Operational Tau method

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Abstract

In this work we present a new approach for the implementation of the operational Tau method for the solution of systems of differential equations.

In this approach the matrix involved in the algebraic equivalent of the differential problem, that is a characteristic of the operational Tau method, has a better condition number, since it is constructed without resource to similarity transformations, hence improving the accuracy of the Tau method solutions.

Keywords: Tau method, Approximation of functions, Conditioning, Orthogonal polynomials

1 Introduction

In this work we present an alternative for the solution of differential problems using an operational version of the Tau method presented by Ortiz and Samara in [11]. This operational version greatly contributed to the development and to generalizations of the Tau method, namely to linear and nonlinear differential problems in several dimensions [9], to systems of linear [5] or nonlinear differential equations in several dimensions [14], to delay differential systems [7] and to integral equations and to fractional differential equations [2], [13], [15].

In the classical operational approach the approximate solution is represented in terms of an orthogonal basis and its constructing process involves the solution of an algebraic linear system of equations, whose matrix is obtained using a similarity transformation. For high degree

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approximations the accuracy of the Tau method is degraded by the bad conditioning of the matrices involved. In our approach we avoid this similarity transformation conferring thus improved accuracy to the Tau method.

Several authors have studied approaches for the Tau method, with the purpose of stabilizing the method, by introducing modifications in the way of representing the solution and thus obtaining a better conditioned algebraic system [4], [6], [12]. Our procedure being more general, unifies and includes the cases cited, with advantages from the point of view of the automation of the algorithms as well as from the numerical point of view.

In Section 2 we review the main results for the classical operational version and we develop the new approach for a solution represented in a general orthogonal polynomial basis. We treat the case of the classical orthogonal polynomial basis in Section 3 and in Section 4 we present results on the conditioning of the algebraic linear system of equations. In Section 5 we review briefly the Tau method. Some illustrative examples are given in Section 6.

2 Algebraic and Analytic Operations on Polynomials

Let \mathcal{D}_v be the class of linear differential operators of order v with polynomial coefficients and $D \in \mathcal{D}_v$

$$D \equiv \sum_{i=0}^v p_i(x) \frac{d^i}{dx^i}, \quad p_i(x) \in \mathbb{P}_{n_i}. \quad (1)$$

The formulation of the operational version of the Tau Method was presented in [11]. In this work, writing $p_i(x) = p_i x$ where $p_i = (p_{i0}, p_{i1}, \dots, p_{in_i}, 0, 0, \dots)$ and $x = (1, x, x^2, \dots)^T$, they prove that the effect of D acting on polynomials, i.e. the effect of differentiation and of multiplication by the variable x on the coefficients $a_n = (a_0, a_1, \dots, a_n, 0, 0, \dots)$ of the polynomial $y_n = a_n x$ can be represented by matrix multiplications. In fact, they show that

$$\frac{d}{dx} y_n(x) = a_n \eta x \quad \text{and} \quad x y_n(x) = a_n \mu x$$

where

$$\eta = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 2 & 0 & \\ & & \dots & \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \dots \end{bmatrix}$$

and, from those relations, the proofs of the main theorems of the operational Tau method follows. In particular, if $D \in \mathcal{D}_v$ is defined like in (1) and $y_n(x) = a_n x$, they show that

$$D y_n(x) = a_n \Pi x \quad \text{where} \quad \Pi = \sum_{i=0}^v \eta^i p_i(\mu) \quad (2)$$

and that if $v = Vx = (P_0(x), P_1(x), \dots)^T$ is a polynomial basis and $y_n(x) = a_n v$ is the expression of y_n in that basis, then

$$Dy_n(x) = a_n \Pi_v v \quad \text{where} \quad \Pi_v = V \Pi V^{-1}. \quad (3)$$

This formulation has the inconvenient of involving a matrix inversion and the process of computing Tau approximants may be unstable. In what follows, we prove that if v is an orthogonal polynomial basis, then from its three-term recurrence relation [1] we can define Π_v avoiding that similarity transformation and so improve the stability in the Tau method computation of approximations of the differential problem solution. We will show that Π_v can be defined from matrices μ_v and η_v such that

$$xv = \mu_v v \quad \text{and} \quad \frac{d}{dx}v = \eta_v v$$

that is, the lines of μ_v and of η_v , are the coefficients vectors of vectors basis v transformation, $xP_j(x)$ and of $\frac{d}{dx}P_j(x)$, represented in basis v .

First of all, with matrices η_v and μ_v , we generalize a lemma from [11]

Lemma 1 *Let $y_n(x) = a_n v \in \mathbb{P}_n$, then*

$$xy_n(x) = a_n \mu_v v \quad \text{and} \quad \frac{d}{dx}y_n(x) = a_n \eta_v v$$

proof.

$$xy_n(x) = x a_n v = a_n x v = a_n \mu_v v$$

and

$$\frac{d}{dx}y_n(x) = a_n \frac{d}{dx}v = a_n \eta_v v \quad \blacksquare$$

Remark 1 *We need not to worry about the meaning of infinite vector and infinite matrix multiplication, since, as we are dealing with polynomials $y_n(x)$, all those products reduce to a finite number of non null parcels. Even when we deal with powers of infinite matrices, like in the next lemma, as they represent polynomial multiplication and differentiation, they have only a finite number of nonzero elements in each line.*

Lemma 2 *For $s, r \in \mathbb{N}$*

$$x^s y_n(x) = a_n \mu_v^s v \quad \text{and} \quad \frac{d^r}{dx^r} y_n(x) = a_n \eta_v^r v$$

proof. *The proof is obvious using induction.* \blacksquare

Now, representing by $q(\mu_v)$ the matrix

$$q(\mu_v) = \sum_{j=0}^m q_j \mu_v^j, \quad \text{if} \quad q(x) = \sum_{j=0}^m q_j x^j$$

and

$$q(\mu_v) = \sum_{j=0}^m q_j P_j(\mu_v), \quad \text{if} \quad q(x) = \sum_{j=0}^m q_j P_j(x)$$

we can represent the effect, in the coefficients a_n , of multiplying $y_n(x) = a_n v$ by $q(x)$.

Lemma 3 Let $q(x) = (q_0, q_1, \dots, q_m, 0, \dots)v$ be a polynomial, then

$$q(x)v = q(\mu_v)v$$

proof. Since $v = (P_0(x), P_1(x), \dots)^T$ is a polynomial basis, then each polynomial $P_j(x)$ has exact degree j , and we can write $P_j(x) = (p_{j,0}, p_{j,1}, \dots, p_{j,j}, 0, \dots)x$, so

$$\begin{aligned} q(x)v &= \sum_{j=0}^m q_j P_j(x)v \\ &= \sum_{j=0}^m q_j \left(\sum_{i=0}^j p_{j,i} x^i v \right) \\ &= \sum_{j=0}^m q_j \left(\sum_{i=0}^j p_{j,i} \mu_v^i \right) v = \sum_{j=0}^m q_j P_j(\mu_v)v = q(\mu_v)v \quad \blacksquare \end{aligned}$$

These previous results are needed to prove the following alternative formulation of the main theorem in [11].

Theorem 1 Let $y_n(x) = a_n v$ and

$$D = \sum_{i=0}^v p_i(x) \frac{d^i}{dx^i}, \in \mathcal{D}_v$$

then

$$Dy_n(x) = a_n \Pi_v v, \quad \text{where} \quad \Pi_v = \sum_{i=0}^v \eta_v^i p_i(\mu_v) \quad (4)$$

proof. Since

$$p_i(x) \frac{d^i}{dx^i} y_n(x) = p_i(x) a_n \eta^i v = a_n \eta^i p_i(x) v = a_n \eta^i p_i(\mu_v) v$$

the result follows. \blacksquare

In order to make this results useful, we need to know how to evaluate the matrices μ_v and η_v . By definition, since v is a polynomial basis, we define an inner product

$$\langle P_i, P_j \rangle = \int_a^b w(x) P_i(x) P_j(x) dx.$$

and so we can evaluate the entries of μ_v and η_v by

$$\mu_{i,j} = \frac{1}{\sqrt{\langle P_i, P_i \rangle}} \langle x P_i, P_j \rangle \quad \text{and} \quad \eta_{i,j} = \frac{1}{\sqrt{\langle P_i, P_i \rangle}} \langle \frac{d}{dx} P_i, P_j \rangle$$

although this is not, in general, the easiest way of evaluate them.

Next we proof that, for the case where v is an orthogonal polynomial basis, this can be achieved from its characteristic three term recurrence relation.

Proposition 1 Let $v = (P_0(x), P_1(x), \dots)^T$ be the orthogonal polynomial basis satisfying the recurrence relation

$$\begin{cases} xP_i(x) = \alpha_i P_{i+1}(x) + \beta_i P_i(x) + \gamma_i P_{i-1}(x), & i \geq 0 \\ P_{-1}(x) = 0, & P_0(x) = 1 \end{cases}$$

Then

$$\mu_v = \begin{bmatrix} \beta_0 & \alpha_0 & & \\ \gamma_1 & \beta_1 & \alpha_1 & \\ & \gamma_2 & \beta_2 & \alpha_2 \\ & & & \dots \end{bmatrix}$$

proof. It is an elementary verification, from the recurrence relation, that

$$x v = (xP_0(x), xP_1(x), \dots)^T = \mu_v v \quad \blacksquare$$

In the following result we will denote $P_i(x)$ by P_i and $\frac{d}{dx}P_i(x)$ by P'_i .

Proposition 2 Let $v = (P_0, P_1, \dots)^T$ be the orthogonal polynomials satisfying the recurrence relation

$$\begin{cases} xP_i = \alpha_i P_{i+1} + \beta_i P_i + \gamma_i P_{i-1}, & i \geq 0 \\ P_{-1} = 0, & P_0 = 1 \end{cases}$$

then

$$\eta_v = \begin{bmatrix} 0 & & & \\ \eta_{10} & 0 & & \\ \eta_{20} & \eta_{21} & 0 & \\ \eta_{30} & \eta_{31} & \eta_{32} & 0 \\ & & & \dots \end{bmatrix}$$

where, for each $i \geq 1$

$$\begin{cases} \eta_{i+1,j} = \frac{1}{\alpha_i}(\alpha_{j-1}\eta_{i,j-1} + (\beta_j - \beta_i)\eta_{i,j} + \gamma_{j+1}\eta_{i,j+1} - \gamma_i\eta_{i-1,j}), & j = 0 : i-1 \\ \eta_{i+1,i} = \frac{1}{\alpha_i}(\alpha_{i-1}\eta_{i,i-1} + 1) \end{cases} \quad (5)$$

proof. By definition, η_v is such that $\frac{d}{dx}v = \eta_v v$, so

$$P'_i = \sum_{j=0}^{i-1} \eta_{i,j} P_j, \quad i \geq 0 \quad (6)$$

Differentiating both sides of the recurrence relation

$$\begin{cases} P_i + xP'_i = \alpha_i P'_{i+1} + \beta_i P'_i + \gamma_i P'_{i-1}, & i \geq 0 \\ P'_{-1} = P'_0 = 0 \end{cases} \quad (7)$$

and replacing (6) in (7), we obtain

$$\alpha_i \sum_{j=0}^i \eta_{i+1,j} P_j = P_i + x \sum_{j=0}^{i-1} \eta_{i,j} P_j - \beta_i \sum_{j=0}^{i-1} \eta_{i,j} P_j - \gamma_i \sum_{j=0}^{i-2} \eta_{i-1,j} P_j \quad (8)$$

and using the recurrence relation again,

$$\begin{aligned}
x \sum_{j=0}^{i-1} \eta_{i,j} P_j &= \sum_{j=0}^{i-1} \eta_{i,j} x P_j \\
&= \sum_{j=0}^{i-1} \eta_{i,j} (\alpha_j P_{j+1} + \beta_j P_j + \gamma_j P_{j-1}) \\
&= \sum_{j=1}^i \alpha_{j-1} \eta_{i,j-1} P_j + \sum_{j=0}^{i-1} \beta_j \eta_{i,j} P_j + \sum_{j=0}^{i-2} \eta_{i,j+1} \gamma_{j+1} P_j
\end{aligned} \tag{9}$$

Substituting (9) in (8) and identifying the coefficients of P_j we get, for $j = 0$,

$$\alpha_i \eta_{i+1,0} = (\beta_0 - \beta_i) \eta_{i,0} + \gamma_1 \eta_{i,1} - \gamma_i \eta_{i-1,0}$$

and, for $j = 1 : i - 2$,

$$\alpha_i \eta_{i+1,j} = \alpha_{j-1} \eta_{i,j-1} + (\beta_j - \beta_i) \eta_{i,j} + \gamma_{j+1} \eta_{i,j+1} - \gamma_i \eta_{i-1,j}$$

This equation also holds for $j = 0$ and for $j = i - 1$ by considering $\eta_{i,j} = 0$ if $j \geq i$ and if $j < 0$, and thus we obtain (5). For $j = i$ the result is quite obvious. ■

We can evaluate η_v using the recurrence relation (5), but first we will show that explicit formulas for the first two subdiagonals of η_v can be obtained.

Proposition 3 *In the conditions of Proposition 2, we have*

$$\eta_{i+1,i} = \frac{i+1}{\alpha_i}, \quad i \geq 0 \tag{10}$$

proof. Since from (7) we have

$$P'_1(x) \equiv \eta_{1,0} = 1/\alpha_0$$

the proof follows by induction over i . ■

Proposition 4 *In the conditions of Proposition 2, we have*

$$\eta_{i+1,i-1} = \frac{1}{\alpha_i \alpha_{i-1}} \left[\sum_{j=0}^{i-1} \beta_j - i \beta_i \right], \quad i \geq 1 \tag{11}$$

proof. By definition

$$P'_2 = \eta_{2,1} P_1 + \eta_{2,0}$$

and writing

$$\begin{aligned}
P'_2 &= \frac{1}{\alpha_1} (P_1 + x P'_1 - \beta_1 P'_1) \\
&= \frac{1}{\alpha_1} (P_1 + \frac{1}{\alpha_0} (\alpha_0 P_1 + \beta_0 - \beta_1)) = \frac{2}{\alpha_1} P_1 + \frac{\beta_0 - \beta_1}{\alpha_1 \alpha_0}
\end{aligned}$$

we derive that $\eta_{2,0} = \frac{\beta_0 - \beta_1}{\alpha_1 \alpha_0}$ and this confirm the result for $i = 1$. From (5) and (10) we can write

$$\eta_{i+1,i-1} = \frac{1}{\alpha_i} \left[\alpha_{i-2} \eta_{i,i-2} + \frac{i}{\alpha_{i-1}} (\beta_{i-1} - \beta_i) \right]$$

and the proof follows by induction. ■

3 Classical Orthogonal Polynomials

In this section we treat the particular cases of the four classical orthogonal polynomial basis. We will find, using their recurrence relations, explicit formulas for the elements of matrices η_v .

Chebyshev Polynomials

$$\begin{cases} xT_i(x) = \frac{1}{2}T_{i+1}(x) + \frac{1}{2}T_{i-1}(x), & i \geq 1 \\ xT_0(x) = T_1(x) \end{cases}$$

and [1]

$$T'_{2i}(x) = 4i \sum_{j=1}^i T_{2j-1}(x), \quad \text{and}, \quad T'_{2i+1}(x) = (2i+1) + 2(2i+1) \sum_{j=1}^i T_{2j}(x).$$

In this case (5) becomes,

$$\begin{cases} \eta_{i+1,j} = \delta_{j,1} \eta_{i,0} + \eta_{i,j-1} + \eta_{i,j+1} - \eta_{i-1,j}, & j = 0 : i-1, i \geq 2 \\ \eta_{i+1,i} = 2(i+1), & \eta_{i+1,i-1} = 0, i \geq 1 \\ \eta_{1,0} = 1 \end{cases}$$

and so $\eta_{i,j} = 0$ except for

$$\begin{cases} \eta_{i,j} = 2i, & j = i-1 : -2 : 1, i \geq 1 \\ \eta_{2i+1,0} = 2i+1, & i \geq 0 \end{cases}$$

and we obtain

$$\eta_T = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 0 & 4 & 0 & & & \\ 3 & 0 & 6 & 0 & & \\ 0 & 8 & 0 & 8 & 0 & \\ 5 & 0 & 10 & 0 & 10 & 0 \\ & & & & & \dots \end{bmatrix} \quad \text{and} \quad \mu_T = \begin{bmatrix} 0 & 1 & & & \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & & \dots & \end{bmatrix}$$

Legendre Polynomials

$$xP_i(x) = \frac{i+1}{2i+1}P_{i+1}(x) + \frac{i}{2i+1}P_{i-1}(x), \quad i \geq 1$$

and

$$P'_{2n}(x) = \sum_{i=1}^n (4i-1)P_{2i-1}(x), \quad \text{and,} \quad P'_{2n+1}(x) = \sum_{i=0}^n (4i+1)P_{2i}(x)$$

Then $\eta_{i,j} = 0$ except for

$$\eta_{i,j} = 2j+1, \quad j = i-1 : -2 : 0, i \geq 1$$

and so

$$\eta_P = \begin{bmatrix} 0 \\ 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 5 & 0 \\ 0 & 3 & 0 & 7 & 0 \\ 1 & 0 & 5 & 0 & 9 & 0 \\ \dots \end{bmatrix} \quad \text{and} \quad \mu_P = \begin{bmatrix} 0 & 1 \\ 1/3 & 0 & 2/3 \\ & 2/5 & 0 & 3/5 \\ & & \dots \end{bmatrix}$$

Laguerre Polynomials

$$xL_i(x) = -(i+1)L_{i+1}(x) + (2i+1)L_i(x) - iL_{i-1}(x), \quad i \geq 0$$

and

$$L'_i(x) = -\sum_{i=0}^{i-1} L_i(x)$$

Then $\eta_{i,j} = 0, \quad j \geq i$

$$\eta_{i,j} = -1, \quad j = 0 : i-1, \quad i \geq 1$$

Then

$$\eta_L = \begin{bmatrix} 0 \\ -1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 \\ \dots \end{bmatrix} \quad \text{and} \quad \mu_L = \begin{bmatrix} 1 & -1 \\ -1 & 3 & -2 \\ & -2 & 5 & -3 \\ & & \dots \end{bmatrix}$$

Hermite Polynomials

$$xH_i(x) = \frac{1}{2}H_{i+1}(x) + iH_{i-1}(x), \quad i \geq 0$$

and

$$H'_i(x) = 2iH_{i-1}(x)$$

Then $\eta_{i,j} = 0, \quad j \neq i-1$ and

$$\eta_{i,i-1} = 2i, \quad i \geq 1$$

and

$$\eta_H = \begin{bmatrix} 0 \\ 2 & 0 \\ & 4 & 0 \\ & & 6 & 0 \\ & & & \dots \end{bmatrix} \quad \text{and} \quad \mu_H = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 & 1/2 \\ & 2 & 0 & 1/2 \\ & & 3 & 0 & 1/2 \\ & & & \dots \end{bmatrix}$$

The explicit computation of the coefficients is more accurate and less expensive than using the recurrence relations. The explicit formulae contribute for the results stability as they are not affected by the errors due to the computations using the recurrence relations.

4 Some numerical details

In this section we analyze the classical Tau method implementation of equation (3) in order to better understand the advantages of our approach. We give bounds on the condition numbers of the matrices involved in the construction of Tau approximants when we use the four classical types of orthogonal polynomials considered in Section 3.

The floating point representation of $\Pi_v = V\Pi V^{-1}$ may be represented as

$$fl(V\Pi V^{-1}) = V\Pi V^{-1} + E \quad (12)$$

where $\|E\|_2 \approx u\kappa_2(V)\|\Pi\|_2$, u is the unit roundoff and $\kappa_2(V)$ is the 2-norm condition number of matrix V [3]. It is also known that $\kappa_2(V) = \frac{\sigma_1}{\sigma_n}$ where σ_1 and σ_n represent, respectively, the maximum and minimum singular values of V . In presence of normal matrices, this can be replaced by the eigenvalue ratio $\frac{|\lambda_1|}{|\lambda_n|}$. Otherwise, the eigenvalues don't have the predictive power of singular values, as it may happen that

$$\max_{i,j} \frac{|\lambda_i|}{|\lambda_j|} \ll \kappa_2(V). \quad (13)$$

In the present case, V denotes the matrix formed by the coefficients of the polynomials in the canonical base. Those are non-normal triangular matrices and so the ratio in equation (13) may be very far from the true value of $\kappa_2(V)$. Nevertheless, it may be regarded as a lower bound of $\kappa_2(V)$ and obtained through $\tilde{\kappa}_2(V) = \max_{i,j} \frac{|v_{ii}|}{|v_{jj}|}$. Based on this and in the explicit formulae for the four classical orthogonal polynomial families [1], we may derive the lower bound estimatives of the condition number of matrix V presented on Table 1.

In Fig. 1 we display the condition number (blue) and the estimative derived from the eigenvalue ratio (red), as explained previously, in terms of the matrix order. A similar condition number in matrix V for both Chebyshev and Legendre polynomials is observed, in opposition to a much bigger condition number for Laguerre or Hermite polynomials. In the same figure, the intersection of the dotted line with the blue line shows that a condition number of order 10^{16} is attained for values of n around 40 for Legendre and Chebyshev, and for a significantly lower value of $n = 18$ for the Laguerre and Hermite polynomials.

In the context of numerical computations, this may have disastrous consequences. Equation (12) leads that, for example, when working with double precision ($u = 2^{-52} \approx 10^{-16}$), the norm of the representation error is $\|E\|_2 \approx 10^{16}\|\Pi\|_2$ for Laguerre polynomials of degree 24, and for degrees around 75 for Chebyshev and Legendre polynomials. We remark that this may explain some of the forthcoming results.

		$\tilde{\kappa}_2(V) = \max_{ij} \frac{ v_{ii} }{ v_{jj} }$
Chebyshev	$T_n(x) = \frac{n}{2} \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}$	2^{n-1}
Legendre	$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}$	$\frac{(2n)!}{2^n * (n! * n!)}$
Laguerre	$L_n(x) = \sum_{m=0}^n (-1)^{n-m} \frac{n! x^{n-m}}{[(n-m)!]^2 m!}$	$n!$
Hermite	$H_n(x) = n! \sum_{s=0}^{[n/2]} (-1)^s \frac{(2x)^{n-2s}}{(n-2s)! s!}$	2^n

Table 1: Lower bounds $\tilde{\kappa}_2(V)$ for the condition number of the matrix V

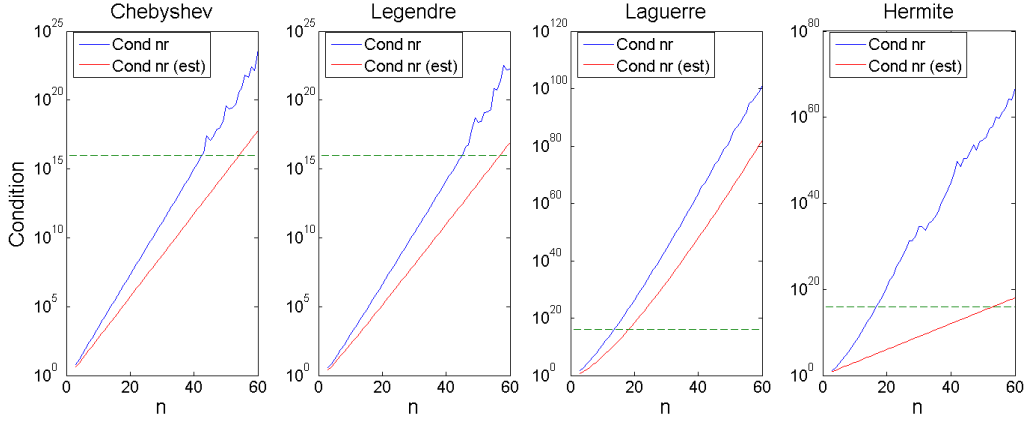


Figure 1: Condition numbers and $\tilde{\kappa}_2(V)$ estimates of matrices V for the four classical orthogonal polynomial families

5 Tau method

A Tau approximant y_n to the solution y of a differential problem

$$\begin{cases} Dy(x) = 0, & x \in \Omega \subset \mathbb{R} \\ D_j y(x) = \sigma_j, & j = 1 : \nu \end{cases} \quad (14)$$

where $D \in \mathcal{D}_\nu$, and $D_j \in \mathcal{D}_\nu$, $j = 1 \dots \nu$ are linear functionals representing supplementary conditions, is the unique polynomial solution of the perturbed problem

$$\begin{cases} Dy_n(x) = \tau(x), & x \in \Omega \\ D_j y_n(x) = \sigma_j, & j = 1 : \nu \end{cases} \quad (15)$$

where τ is a polynomial perturbation close to zero in Ω , [10]. In the sense of the Tau method this means that the approximant y_n satisfies the supplementary conditions and is such that Dy_n coincides with Dy as far as possible.

As shown in Section 2, in terms of the operational Tau method, $y_n = a_n v$ is the solution of

$$a_n \Gamma_\nu = [\sigma, 0, 0, \dots]$$

where $\Gamma_\nu = [B_\nu : \Pi_\nu]$, B_ν is the matrix representing the operators in the supplementary conditions, $\sigma = [\sigma_1, \dots, \sigma_\nu]$ and Π_ν is given by (4).

The matrix Γ_ν is slightly better conditioned than matrix Π_ν , but in our numerical experiments we observed that it inherits the bad conditioning properties of Π_ν .

6 Numerical results

We will solve some problems for which we know the exact solutions in order to test our procedure and compare it, in terms of the absolute error, with other methods.

For these problems, we compare numerical results obtained with the two approaches described in this work for the computation of matrix Π . Using our procedure, we compute matrix Π^D directly from the three-term recurrence relation for the orthogonal polynomials, and using the classical approach, computing $\Pi^{ST} = V\Pi V^{-1}$ via the similarity transformation.

6.1 Example 1

We consider a boundary value problem from [6], for which the coefficients of the expansion of the exact solution in Chebyshev series are known. This problem allows the comparison with Tau approximants obtained using other techniques, reported by the authors in their paper.

Example 1

$$\begin{cases} y'''(x) - k^2 y'(x) + 1 = 0, & x \in]0, 1[\\ y'(0) = y'(1) = 0, & y(\frac{1}{2}) = 0 \end{cases}$$

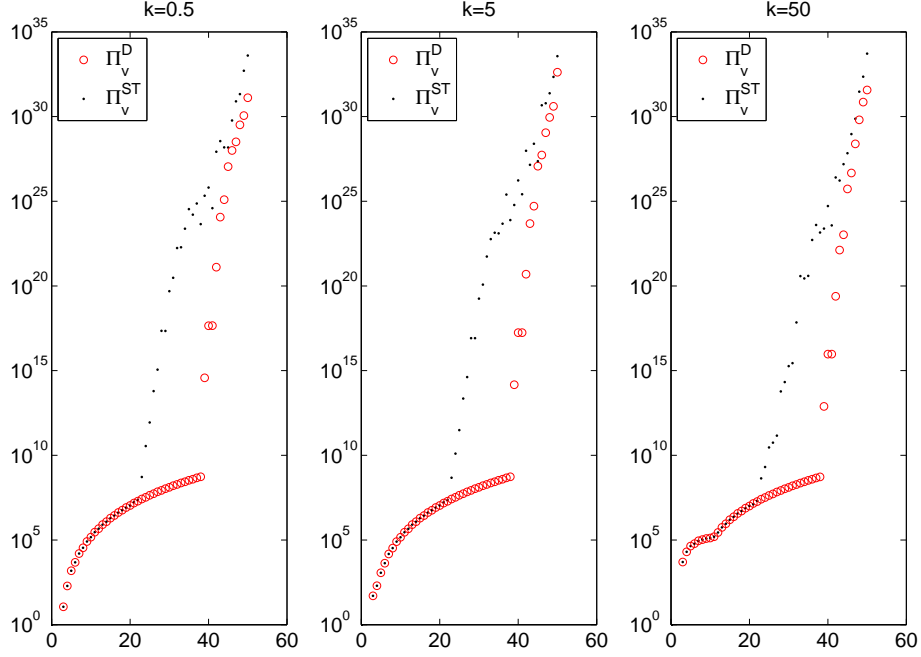


Figure 2: Condition numbers of Γ^D using our procedure, and Γ^{ST} using singularity transformation, in Example 1, using Chebyshev polynomials, for $k = 0.5, 5, 50$ with n -degree polynomials $n = 3 : 50$

For this case the matrix corresponding to the algebraic system (4) is given by

$$\Pi_v^D = \eta_v^3 - k^2 \eta_v.$$

In [6] the authors transform the differential equation in a system of 3 equations and, in order to obtain a Tau approximant y_n , they solve an algebraic system of order $(3(n+1)) \times (3(n+1))$ using an iterative method. The best numerical results reported, for $k = 5$, have maximum error 1.3×10^{-15} , after 17 iteration steps, and 2.3×10^{-14} , after 16 iteration steps, for $n = 16$ and $n = 32$, respectively. In Fig. 2 we show that they are not able to go further due to the fast increasing, with n , of the condition number of the matrices Γ_v , and that with our procedure we slow this increasing, allowing the computation of approximants with larger n . Another advantage of our approach is that to obtain y_n we have a system of order $n \times n$ that we solve by a direct method.

In Fig. 3 we compare, in terms of the absolute errors $|y(x) - y_n(x)|$, the approximation obtained with our matrices Π^D , with the approximation obtained with the matrices Π^{ST} , built with the similarity transformation, both for Chebyshev polynomials basis. As expected for small degree polynomials ($n \leq 16$), we get the same results with both procedures. As n increases, the similarity transformation procedures fails, while our approach proceeds up to $n = 37$ without failure in the approximation.

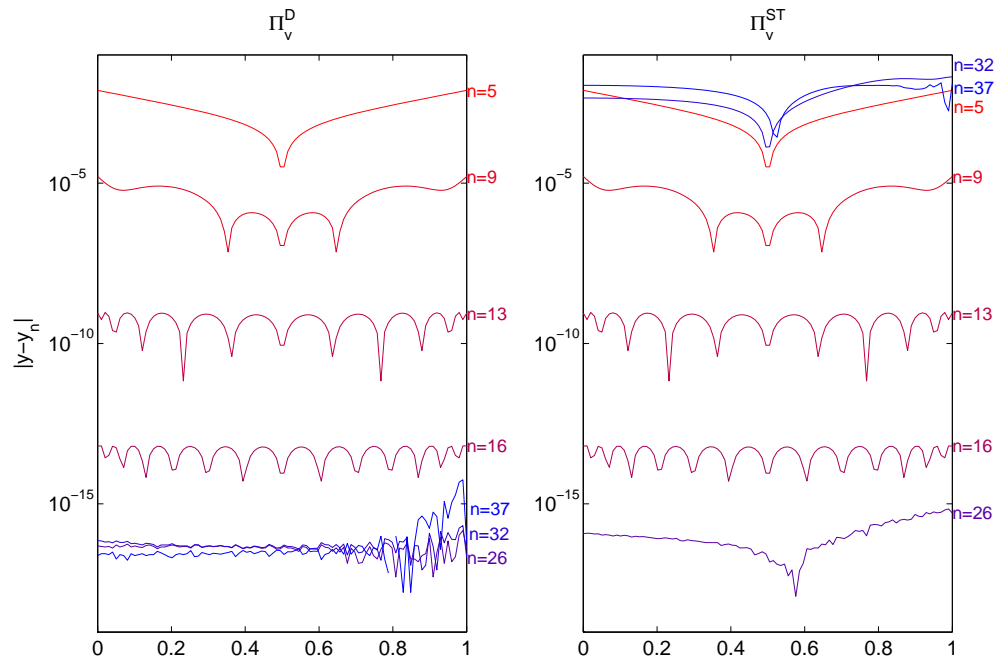


Figure 3: Absolute errors $|y(x) - y_n(x)|$, $x \in]0, 1[$ for selected n -degree polynomials and for $k=5$, in Example 1, using for the computation of matrix Π , our procedure, Π^D , and the singularity transformation Π^{ST}

6.2 Example 2

We consider an initial value problem in $[-1, 1]$, depending on a parameter $s \geq -1$ that introduces a singularity in the solution near the orthogonality interval. So only for large values of n will we obtain good approximations of the solution.

Example 2

$$\begin{cases} 2(x+s)y'(x) - y(x) = 0, & x \in]-1, 1[, & s \geq 1 \\ y(-1) = 1 \end{cases}$$

whose solution is

$$y(x) = \sqrt{(x+s)/(s-1)}$$

then

$$\Pi_v^D = 2\eta_v(\mu_v + sI) - I$$

and we will choose distinct values of s and distinct values of n .

In Fig. 4 we can see the same increasing values of the condition numbers of Γ matrices, in example 2, that we have observed in example 1. Also, we can observe that the increasing behavior of the condition number is not greatly influenced by the parameter s .

By other way, we can observe in Fig. 5, for three distinct values of s , that the quality of the approximation is greatly influenced by s , at least near $s = 1$. This behavior is also observed in Fig. 6 where we show that our procedure allows the computation of higher degree approximants of the solution without significant loss of accuracy.

7 Conclusions

In this paper we have presented a new implementation for the operational Tau method where Π_v is defined without resource to similarity transformations, at least when dealing with an orthogonal polynomial basis, with both numerical and computational advantages over the classical formulation of Ortiz and Samara, that has been used in the generalizations of the Tau method.

The numerical results presented were obtained using the Chebyshev polynomial basis, but our procedure is exactly the same for any other orthogonal polynomials basis. We have solved the examples presented using Legendre polynomials that we do not include since we got similar results.

The numerical results showed that we get more accurate Tau approximants, for large values of the order of the approximation n , than with other implementations and other methods of computation of the approximants.

This fact is of great importance when there is need of a large number of coefficients computed with great precision, as it is in the case with Frobenius-Padé approximation allowing the computation of rational approximants of series with unknown coefficients [8].

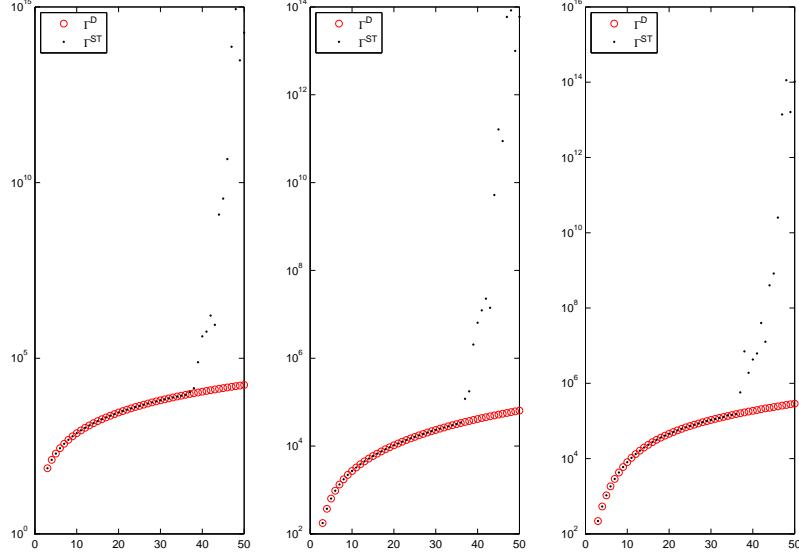


Figure 4: Condition numbers of Γ^D using our procedure, and Γ^{ST} using singularity transformation, in Example 2, using Chebyshev polynomials, for $y = \sqrt{5x+6}$ with $s = 1.2$, for $y = \sqrt{50x+51}$ with $s = 1.02$ and for $y = \sqrt{500x+501}$ with $s = 1.002$

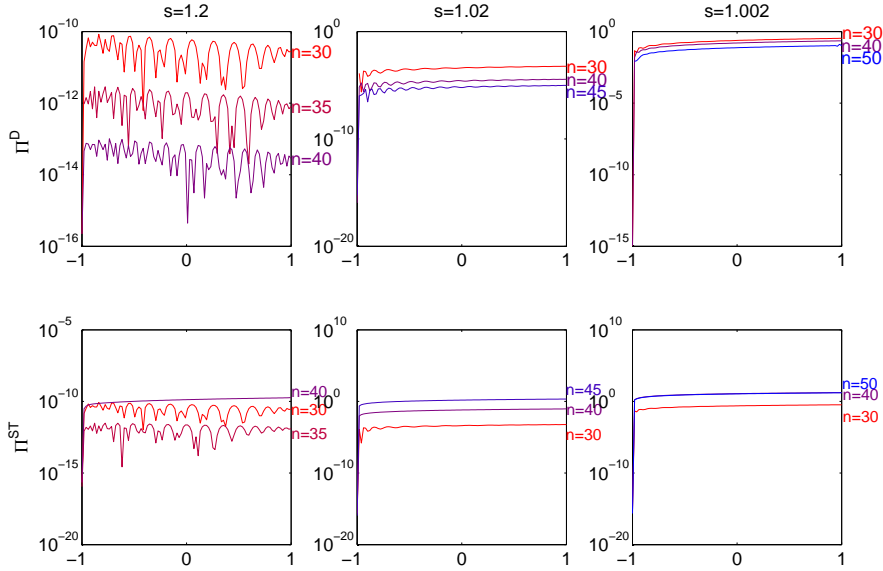


Figure 5: Absolute error curves of Tau approximants y_n , for $y = \sqrt{5x+6}$ with $s = 1.2$, for $y = \sqrt{50x+51}$ with $s = 1.02$, for $y = \sqrt{500x+501}$ with $s = 1.002$, using for the computation of matrix Π , our procedure, Π^D , and the singularity transformation Π^{ST}

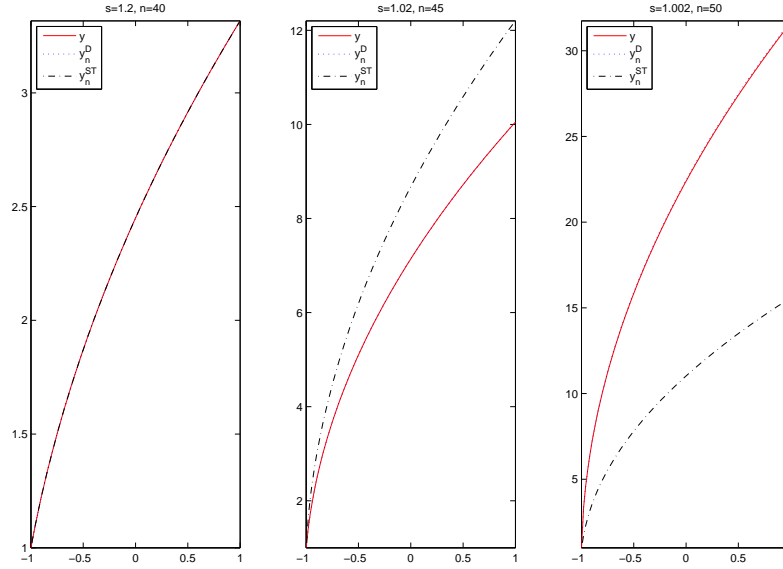


Figure 6: Plots of $y(x)$, $y_n^D(x)$ and $y_n^{ST}(x)$, $x \in]-1, 1[$, using for the computation of matrix Π , our procedure and the singularity transformation, respectively

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